§ Differential Forms

- (Exterior Linear Algebra)

Goal: V n-dim'l vector space $\rightsquigarrow \Lambda^k V^*$

Method 1: $(k=2)$ $\quad v_1^* \wedge v_2^* = - v_2^* \wedge v_1^*$ $\quad$ (E.g. $v \wedge v = 0$)

Method 2: Consider the "skew-symmetrization" operator

$$\mathcal{A} : \otimes^k V^* \longrightarrow \otimes^k V^* .$$

$$\mathcal{A}(\alpha)(v_1, \dots, v_k) = \sum_{\sigma \in \mathcal{S}_k} \operatorname{sign}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

E.g.) $\mathcal{A}(v_1^* \otimes v_2^*) = v_1^* \otimes v_2^* - v_2^* \otimes v_1^*$

Def$^{\underline{n}}$: $\alpha \in \otimes^k V^*$ is skew-symmetric if $\mathcal{A}(\alpha) = \alpha$

Denote: $\quad \Lambda^k V^* := \{ \alpha \in \otimes^k V^* \text{ skew-symmetric} \} \subseteq \otimes^k V^*$

"Wedge/exterior Product"

$$\wedge : \Lambda^k V^* \times \Lambda^l V^* \longrightarrow \Lambda^{k+l} V^*$$

$$\alpha \wedge \beta := \frac{1}{k!\,l!} \mathcal{A}(\alpha \otimes \beta)$$

Exterior Algebra: $\quad \Lambda^* V^*$

E.g.) $\alpha, \beta \in \Lambda^1 V^* \cong V^*$

$$\Rightarrow \quad (\alpha \wedge \beta)(u, v) = \det \begin{pmatrix} \alpha(u) & \beta(u) \\ \alpha(v) & \beta(v) \end{pmatrix}$$

Properties: (1) $\alpha\wedge\beta = (-1)^{k\ell}\beta\wedge\alpha \quad \forall\, \alpha\in\Lambda^k V^*,\ \beta\in\Lambda^\ell V^*$

(2) $\dim \Lambda^k V^* = \binom{\dim V}{k}$ E.g.) $\Lambda^0 V^* \cong \Lambda^n V^* \cong \mathbb{R}$

$\Lambda^1 V^* \cong V^*$; $\Lambda^k V^* = 0$ when $k>n$

Why? $\left.\begin{array}{l}\{e_1,\dots,e_n\} \text{ basis for } V \\ \{e_1^*,\dots,e_n^*\} \text{ dual basis for } V^*\end{array}\right\} \Rightarrow \{e_{i_1}^*\wedge\dots\wedge e_{i_k}^*\}_{1\le i_1<\dots<i_k\le n}$ basis for $\Lambda^k V^*$.

- Apply them to $V = T_pM$, we obtain

Exterior Bundle: $\Lambda^k T^*M := \prod_{p\in M} \Lambda^k T_p^*M$

$$\Gamma(\Lambda^k T^*M) := \{k\text{-forms on } M\} =: \Omega^k(M)$$

Properties: (i) $\forall\, \phi\in \mathrm{Diff}(M)$, $\phi^*(\alpha\wedge\beta) = (\phi^*\alpha)\wedge(\phi^*\beta)$.

(ii) $\mathcal{L}_X(\alpha\wedge\beta) = (\mathcal{L}_X\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_X\beta)$.

Exterior Derivative:

$\exists$ linear operator $d:\Omega^k(M)\to\Omega^{k+1}(M)$ s.t.

(i) $k=0$: $(\Omega^0(M) = C^\infty(M))$

$d: C^\infty(M)\to\Omega^1(M)$ agrees with df, $\forall f\in C^\infty(M)$

ie $(df)(X) = X(f) \quad \forall f\in C^\infty(M)$

(ii) $d(df) = 0 \quad \forall f\in C^\infty(M)$ ie $d^2 = 0$ on $\Omega^0(M)$.

(iii) $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^k\alpha\wedge(d\beta)$, $\forall \alpha\in\Omega^k,\ \beta\in\Omega^\ell$

<u>FACT</u>: Such an operator exists & is uniquely defined by (i) – (iii).

<u>Reason</u>: locally. $\alpha = \sum_{I=(i_1,\dots,i_k)} \alpha_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(M)$

$$d\alpha = \sum_I d\left(\alpha_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k}\right)$$

$$= \sum_I d\alpha_I \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} + \underbrace{\left(\alpha_I \, \cancelto{0}{d^2x^{i_1}} \wedge \dots \wedge dx^{i_k} + 0\right)}_{=0}$$

$$= \sum_I \sum_l \frac{\partial \alpha_I}{\partial x^l} dx^l \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

<u>E.g.</u>) $d(x^2 dy \wedge dz) = d(x^2) \wedge dy \wedge dz + 0$

$$= \left[\left(\frac{\partial}{\partial x} x^2\right) dx + \left(\frac{\partial}{\partial y} x^2\right) dy + \left(\frac{\partial}{\partial z} x^2\right) dz\right] \wedge dy \wedge dz$$

$$= 2x \, dx \wedge dy \wedge dz.$$

<u>Properties of d</u>:

(a) $d^2 = 0$ on $\Omega^k(M)$ <u>for all k</u>

<u>Digression</u>: de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

Now, $d^2 = 0 \Rightarrow \ker(d) \supseteq \operatorname{Im}(d)$

defines de Rham cohomology: $H^k_{dR}(M) := \dfrac{\ker(d)}{\operatorname{Im}(d)}$

(b) $d \circ \phi^* = \phi^* \circ d \quad \forall \phi \in \mathrm{Diff}(M).$

and $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d \quad \forall X \in \Gamma(TM).$

(c) <u>Cartan's "Magic" Formula</u>

$$\boxed{\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d} \quad \text{on } \Omega^k(M).$$

"Proof": (a) - (b) easy to check.

(c) Check $P_X := d \circ \iota_X + \iota_X \circ d$ satisfies the properties of L_X

E.g.) $P_X(f) = d \circ \underbrace{\cancel{\iota_X(f)}}_{0} + \iota_X \circ \overbrace{d(f)}^{df} = df(X) = X(f) = L_X f$

$\square$

Cor: $\forall\, \alpha \in \Omega^1(M)$,

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

Pf: Apply Cartan's formula to 1-forms:

$$(L_X\alpha)(Y) = X(\alpha(Y)) - \alpha(L_X Y) = X(\alpha(Y)) - \alpha([X,Y])$$

$\Vert$ Cartan

$$(d\circ\iota_X\alpha + \iota_X\circ d\alpha)(Y) = \underbrace{d(\alpha(X))(Y)}_{Y(\alpha(X))} + d\alpha(X,Y)$$

$\square$

§ Volume Forms & Integration

Def$^{\underline{n}}$: A volume form on M^n is a nowhere vanishing n-form ω

i.e. $\omega \in \Omega^n(M)$, $\omega_p \neq 0$ at each $p \in M$.

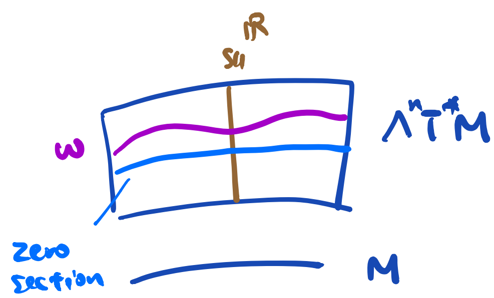
Thm: TFAE:

(i) $\exists$ volume form ω on M^n

(ii) $\Lambda^n T^*M$ is a trivial (rank 1) bundle over M

(iii) M is orientable.

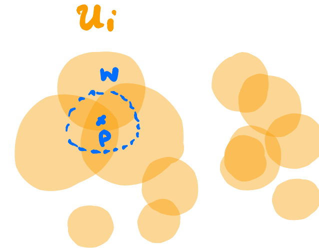
"Sketch of Proof": (i) $\Leftrightarrow$ (ii) trivial



(i) $\Rightarrow$ (iii)

locally. $\omega = \overset{\neq 0}{a}\, dx^1\wedge\cdots\wedge dx^n$

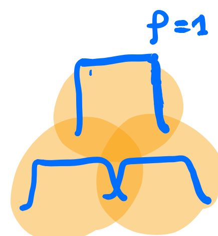
$\left(\begin{array}{l} F: \mathbb{R}^n \to \mathbb{R}^n \text{ diffeo.} \\ F^*(dx^1\wedge\cdots\wedge dx^n) = \det(dF)\, dx^1\wedge\cdots\wedge dx^n \end{array} \right)$



(iii) $\Rightarrow$ (i) Use "Partition of Unity"

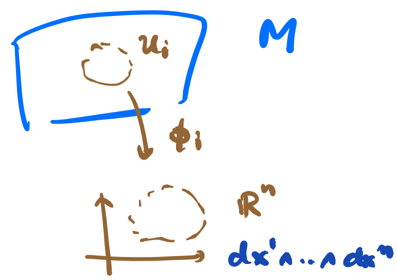
$\{\mathcal{U}_i\}_{i\in I}$ "locally finite" open cover of M
ie $\forall p\in M$, $\exists$ nbd. $W\subseteq M$ of p st $W\cap\mathcal{U}_i \neq \phi$ for finitely many $i\in I$

A partition of unity subordinate to $\{\mathcal{U}_i\}_{i\in I}$ is a family of smooth functions $\rho_i : M \to \mathbb{R}_{\geq 0}$, $i\in I$, s.t.



- $\operatorname{supp} \rho_i \subseteq \mathcal{U}_i \quad \forall i \in I$
- $\underbrace{\sum_{i\in I} \rho_i(p)}_{\text{finite sum}} = 1 \quad \forall p \in M.$

Locally, $dx^1\wedge\cdots\wedge dx^n$ is a volume form on $\mathbb{R}^n$ (loc. defined on M)



$\{(\mathcal{U}_i, \phi_i)\}_{i\in I}$ loc. finite oriented atlas

$\rightsquigarrow$ $\{\rho_i\}_{i\in I}$ partition of unity

$$\omega := \sum_{i\in I} \rho_i\, \phi_i^*(dx^1\wedge\cdots\wedge dx^n)$$

is a volume form.

Integration : on oriented M $\quad \int_M : \Omega^n(M) \to \mathbb{R}$; $\int_M \alpha := \sum_{i\in I} \int_{\mathcal{U}_i} \rho_i \alpha = \sum_{i\in I} \underbrace{\int_{\mathcal{U}_i} (\rho_i\circ\phi_i^{-1})(\phi_i^{-1})^*\alpha}_{\text{integration on } \mathbb{R}^n}$

Fixing volume form ω $\Rightarrow$ $\int_M : C^\infty(M) \to \mathbb{R}$: $\int_M f := \int_M f\omega$